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FACTORIZATION IN DEDEKIND DOMAINS WITH FINITE CLASS GROUP

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ABSTRACT

Let D be a Dedekind domain. It is well known that D is then an atomic integral domain (that is to say, a domain in which each nonzero nonunit has a factorization as a product of irreducible elements). We study factorization properties of elements in Dedekind domains with finite class group. If D has the property that any factorization of an element α into irreducibles has the same length, then D is called a half factorial domain (HFD, see [41]). If D has the property that any factorization of an element α into irreducibles has the same length modulo r (for some $r > 1$), then D is called a congruence half factorial domain of order r. In Section I we consider some general factorization properties of atomic integral domains as well as the interrelationship of the HFD and CHFD property in the Dedekind setting. In Section II we extend many of the results of [41], [42] and [36] concerning HFDs when the class group of D is cyclic. Finally, in Section III we consider the CHFD property in detail and determine some basic properties of Dedekind CHFDs. If G is any Abelian group and S any subset of $G - \{0\}$, then $\{G, S\}$ is called a realizable pair if there exists a Dedekind domain D with class group G such that S is the set of nonprincipal classes of G which contain prime ideals. We prove that for a finite abelian group G there exists a realizable pair $\{G, S\}$ such that any Dedekind domain associated to $\{G, S\}$ is CHFD for some $r > 1$ but not HFD if and only if G is not isomorphic to $\mathbf{Z}_2$, $\mathbf{Z}_3$, $\mathbf{Z}_2 \oplus \mathbf{Z}_2$, or $\mathbf{Z}_3 \oplus \mathbf{Z}_3$.

I. Introduction

The study of unique factorization domains (UFDs) has played an important role in commutative algebra for many years. The papers [6] and [32]–[34] outline some of the major results in this area. If D is a Dedekind domain it is well known

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that D is UFD if and only if the ideal class group of D is trivial. In the case where D is a ring of algebraic integers, it is widely acknowledged that the size of the ideal class group is a measure as to exactly how far D is from having unique factorization. Carlitz was the first author to attach an arithmetical characterization to the above idea. The main result of his paper [1], whose proof relies on the fact proved in [14] that each ideal class of an algebraic number ring contains a prime ideal, is as follows:

THEOREM. *Let D be the ring of integers in a finite extension $K = \mathbf{Q}[\theta]$ of the rationals. Then K has class number less than or equal to two if and only if every factorization of an integer α into irreducible factors contains the same number of irreducible factors.*

The above characterization has led to much additional research in commutative algebra and algebraic number theory. The objective of this work is to expand the existing results in these areas and explore in greater detail factorization properties in Dedekind domains with finite class group. We concentrate on this case since it parallels the situation in the ring of integers of a classic number field. Since the result cited above in [14] does not hold in a general Dedekind domain, these problems present new difficulty. We begin with a brief discussion of the known results concerning factorization in algebraic number rings and then concentrate more closely on the general Dedekind case. The proofs of several of these results involve an interesting combination of concepts from number theory, commutative algebra, group theory and even combinatorics. For readers unfamilar with ideal theory in the general Dedekind setting, Gilmer [11] is a good reference.

Let D be the ring of integers in a finite algebraic number field with class number h. In [21] Narkiewicz poses the problem of giving an arithmetic characterization of algebraic number rings with $h \neq 1$ or 2 along the lines of Carlitz's Theorem above. The papers of Czogala [7], Krause [16], DiFranco and Pace [8], Kaczorowski [15], and Salce and Zanardo [31] all address this problem. In [30] Rush solved Narkiewicz's problem using sets of completely irreducible elements (see [15]). The papers [7] and [31] consider the following conditions on D which will be of interest in Section II. Let n and k be positive integers greater than one and $\alpha_1, \alpha_2, \beta_1, \ldots, \beta_{k-1}$, and β_k any irreducibles of D. D has property V_n if the equality $\alpha_1\alpha_2 = \beta_1 \cdots \beta_k$ implies $k \leq n$. D has property W_n if it has property V_n and $\alpha_1^2 = \beta_1 \cdots \beta_k$ implies $k = 2$. Czogala observes in [7] that D has class number less than or equal to two if and only if D has property V_2.

Suppose D is as above and $h > 2$. The works of Narkiewicz [18]–[27], Sliwa [37][38] and Geroldinger [10] all consider specific problems concerning factorization properties of such algebraic number rings. In particular, Sliwa shows in [38] that if G_m is the set of algebraic integers in D with factorizations of m different lengths, then $G_m \neq \emptyset$ for all positive integers m. Geroldinger [10] generalizes Sliwa's result in the following manner. Let α be a nonzero nonunit of D and set $L(\alpha) = \{m \mid \alpha$ has an irreducible factorization containing m elements$\}$. Geroldinger shows that for a domain D with divisor theory (see [35]) the set $L(\alpha)$ is arithmetic for almost all $\alpha \in D$. For rings of algebraic integers he shows that for all nonzero nonunits α of D

$$L(\alpha) = \{m, m+1, \ldots, m+k\}$$

for some nonnegative integers m and k which depend on α. Many of the results of Sliwa and Geroldinger are dependent on properties of finite abelian groups studied in [22], [27], [28] and [29].

In [21] and [26] Narkiewicz poses the question of characterizing Dedekind domains which satisfy the factorization property stated in Carlitz's Theorem. The papers of Zaks [41][42] consider this problem in broader generality in an *atomic domain* (an integral domain D is atomic if any nonzero nonunit α of D can be written as a product of irreducible elements; see [5], [12] and [40]). In [42] Zaks defines an atomic integral domain which satisfies the factorization property of the Carlitz Theorem to be a *half factorial domain* (*HFD*). Papers by Skula [36], Michel and Steffan [17] and Steffan [39] also consider integral domains with this property. Notice that if D and K are as in [1] then Carlitz's Theorem may be restated as follows: a finite algebraic number field K has class number less than or equal to two if and only if D is HFD. In [42] Zaks also makes the following definition which is crucial to the characterization of Dedekind HFDs. Let $P = \{p_1, \ldots, p_k\}$ be a set of positive integers. P is splittable if whenever $(n_1/p_1) + \cdots + (n_k/p_k)$ equals an integer for $n_1, \ldots, n_k$ positive integers there exist nonnegative integers $m_1, \ldots, m_k$ such that $m_i \leq n_i$ for all i and $(m_1/p_1) + \cdots + (m_k/p_k) = 1$. Skula [36] refers to such a set as a C-set. We note that not much is known in general about splittable sets of positive integers. Zaks' characterization which appears in [42] is as follows (another form of this result appears in [36]).

THEOREM. *Let D be a Dedekind domain with torsion class group G. Then D is HFD if and only if whenever $P_1, \ldots, P_t$ are prime ideals so that $P_1^{n_1} \ldots P_t^{n_t} = Dx$ then there exists a subproduct $P_1^{m_1} \ldots P_t^{m_t} = Dy$, $m_i \leq n_i$, where y is an irreducible such that $(m_1 | s_1) + \ldots + (m_t | s_t) = 1$ where s_i is the order of P_i.*

Zaks and Skula also prove several other important results concerning HFDs in the papers cited above. These include: (1) If G is a nontrivial finite abelian group, then there exists a Dedekind domain D with class group G such that D is HFD. (2) If D is Dedekind with class group $\mathbf{Z}_p$ for p a prime integer, then D is HFD if and only if all the nonprincipal prime ideals of D are in one ideal class. (3) Let D be an atomic integral domain and let M be the set of nonzero nonunits of D. D is HFD if and only if there exists a function $l:\Omega \to \mathbf{Z}^+$ such that the image of Ω under l is $\mathbf{Z}^+$, $l(st) = l(s) + l(t)$ for all s and t in Ω, and $l(s) = 1$ if and only if s is irreducible in D. The function l is called a *length function* on D. In [41], Zaks asks whether (1) above generalizes to any abelian group G. Michel and Steffan [17] extend (1) to three new classes of abelian groups: torsion groups, free groups, and divisible groups. Whether or not the result holds for all abelian groups G is still open.

Consider now the following generalization of a half factorial domain defined by the present authors in [2]. An atomic integral domain D is a *congruence half factorial domain (CHFD) of order r* if and only if there exists an integer $r > 1$ such that if $\prod_{i=1}^{n} x_i = \prod_{j=1}^{m} y_j$ with all the x_i and y_j irreducible, then $n \equiv m \pmod{r}$. Clearly HFD implies CHFD for all $r > 1$. The results of [2] are twofold. First, the authors construct a class of Dedekind domains which are CHFD for some $r > 1$ but not HFD. Second, they show that Carlitz' Theorem can be restated in the following manner: if D is the ring of integers of a finite algebraic extension $K = \mathbf{Q}[\theta]$ of the rationals, then K has class number less than or equal to two if and only if D is CHFD for some $r > 1$.

Before continuing we consider a result important to extending the ideas discussed to this point. Let G be an abelian group and S any subset of $G - \{0\}$. Call the pair $\{G, S\}$ *realizable* [17] if there exists a Dedekind domain D with class group G such that the set of nonprincipal ideal classes which contain prime ideals is precisely S. The question of when the pair $\{G, S\}$ is realizable is explored in papers by Grams [13], Michael and Steffan [17], and Skula [36]. Grams wrote the earliest paper to completely answer the question for finite groups. Her work is based on several important results of Claborn [3][4] which are also summarized in [9]. The major result is as follows:

THEOREM. *Let G be a finite abelian group and S a subset of $G - \{0\}$. The pair $\{G, S\}$ is realizable if and only if S generates G.*

The results presented in this paper are in three areas. In Section II we consider the interrelationship between atomic, half factorial and congruence half factorial domains. We introduce the notion of representing nonprime irreducible elements of a Dedekind domain as vectors and define a closure property on the set of irreducible vectors of D which is related to the properties V_n and W_n discussed by Czogala in [7]. We draw further parallels between closure properties of these vectors and counting the largest number of factorizations of different lengths that a product of n irreducibles might have in D. In Section III we apply the vector terminology of Section II and Grams' Theorem to the existing work of Zaks on HFDs to obtain a better description of realizable pairs $\{\mathbf{Z}_n, S\}$ with associated Dedekind domains which are HFD. In particular, when the elements of S all divide n we introduce a characterization alternate to that of splittability. In Section IV we expand our work in [2] and discuss the basic properties of CHFDs. Finally, we prove the following Theorem: for a finite Abelian group G there exists a subset S of $G - \{0\}$ such that any Dedekind domain associated to the realizable pair $\{G, S\}$ is CHFD of order r for some $r > 1$ but not HFD if and only if G is not one of the groups $\mathbf{Z}_2$, $\mathbf{Z}_3$, $\mathbf{Z}_2 \oplus \mathbf{Z}_2$, or $\mathbf{Z}_3 \oplus \mathbf{Z}_3$.

In questions related to the counting of the number of irreducible factors in factorizations in a Dedekind domain, those irreducibles which generate principal prime ideals always behave well due to the unique factorization of ideals in the Dedekind setting. That is, if $\gamma = \alpha_1 \cdots \alpha_k = \beta_1 \cdots \beta_t$ represents two factorizations of γ into irreducibles, the number of α_i which generate principal primes is the same as the number of β_i which generate principal primes. Thus for the following discussion we will usually assume without loss that the factorizations are produced only by irreducibles that do not generate prime ideals.

II. Atomic domains and closure properties

We can immediately deduce the following relationship between HFDs and CHFDs.

THEOREM 2.1. *Let D be an atomic integral domain. The following are equivalent:*

(1) *D is HFD.*

(2) *D is CHFD of order r for all $r > 1$.*

(3) *D is CHFD of order r for infinitely many positive integers r.* ■

Before exploring Dedekind HFDs and CHFDs in detail, we consider the HFD, CHFD and several related conditions on an atomic domain in general. In [5], Cohn states without proof that an atomic integral domain satisfies the ascending chain condition for principal ideals (a.c.c.p.). Grams [12] provided a counterexample to this conjecture. Zaks [40] provided several additional examples of atomic domains without the a.c.c.p. We now show a positive relationship involving the HFD and a.c.c.p. properties.

THEOREM 2.2. *Let D be HFD. Then D satisfies the a.c.c.p.*

PROOF. Let D be HFD and suppose x_1 and x_2 are nonzero nonunits of D. Let $l(x)$ be the length function on D mentioned in Section I. Then $(x_n) \subseteq (x_m)$ implies $l(x_m) \leq l(x_n)$. It easily follows that if $x_n \in (x_m)$ and $l(x_n) = l(x_m)$ then $(x_n) = (x_m)$. Thus if

$$(x_1) \subseteq (x_2) \subseteq \cdots \subseteq (x_n) \subseteq \cdots$$

were an ascending chain of proper principal ideals, it becomes stationary when $l(x_n)$ attains its minimum. ■

The above theorem leads us to the following general relationships:

THEOREM 2.3. *Let D be an integral domain. Consider the conditions:*
(1) *D is UFD.*
(2) *D is HFD.*
(3) *D satisfies the a.c.c.p.*
(4) *D is atomic.*
The following implications, and no others, hold:

$$(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4).$$

PROOF. Clearly (1) implies (2). (2) implies (3) by the Theorem above. (3) implies (4) by an elementary argument. The domain constructed by Grams in [12] provides a counterexample to (4) implies (2), (3) and (1). Any ring of algebraic integers of class number greater than 2 provides a counterexample to (3) implies (2) and (1). Any ring of algebraic integers of class number 2 shows that (2) does not imply (1). ■

If
(5) *D is CHFD for some r > 1,*
then the diagram above becomes

$$\begin{array}{c}(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4)\\ \Downarrow \\ (5) \\ \Downarrow \\ (4)\end{array}$$

The question whether or not (5) ⇒ (3) remains open.

We introduce some terminology. Let D be a Dedekind domain with realizable pair $\{C(D),\Lambda\}$ where $\Lambda=\{\lambda_\alpha\}_{\alpha\in\mathfrak{A}}$ and $\mathfrak{A}$ is some indexing set. By unique factorization of ideals into finite products in D we have for any ideal J of D a finite subset $\Lambda_J=\{\lambda_{\alpha_1},\ldots,\lambda_{\alpha_{k_J}}\}$ of distinct elements of Λ such that J can be factored as

$$(1)\qquad J=P_{\alpha_1,1}\cdots P_{\alpha_1,x_1}P_{\alpha_2,1}\cdots P_{\alpha_2,x_2}\cdots P_{\alpha_{k_J},1}\cdots P_{\alpha_{k_J},x_{k_J}}$$

where each $P_{\alpha_i,t}$ is a prime ideal of class λ_{α_i} for i and t such that $1\le i\le k_J$ and $1\le t\le x_i$. Now suppose $J=(\gamma)$ for some γ in D and let $\bar{I}$ represent the ideal class of an ideal I of D. Then there exists a subset $\Lambda_{(\gamma)}=\{\lambda_{\alpha_1},\ldots,\lambda_{\alpha_{k(\gamma)}}\}$ of Λ such that

$$\overline{(\gamma)}=\bar{P}_{\lambda_1}^{x_1}\cdots\bar{P}_{\lambda_{k(\gamma)}}^{x_{k(\gamma)}}$$

for prime ideals $\{P_{\alpha_i}\}_{i=1}^{k(\gamma)}$ each of class λ_{α_i} in Λ. Now, set F to be the free group on Λ. Let

$$(2)\qquad v=\sum_{\lambda\in\Lambda}x_\lambda\cdot\lambda$$

where $x_\lambda=0$ if $\lambda\notin\Lambda_{(\gamma)}$ and $x_\lambda=x_i$ when $\lambda=\lambda_{\alpha_i}$ for some $1\le i\le k_{(\gamma)}$. Since v is in F we have a map from the principal ideals generated by elements of D which extends in the usual manner to a group homomorphism

$$\Psi:(\mathrm{Prin}(D)\setminus(0))\to F$$

from the group of all nonzero principal ideals of D to F. Thus for each nonzero $\gamma\in D$ we can associate a vector v in F. We will use the notation $v=\langle\langle x_\lambda\rangle\rangle_{\lambda\in\Lambda}$ to denote such vectors. If γ is irreducible, we will say that its associated vector v is of *irreducible type.* It is easy to see that the map Ψ is well defined with respect to ideals generated by irreducible elements (that is, if γ is irreducible in D and $\Psi((\gamma))=v$ then $\Psi((\alpha))=v$ implies that α is also irreducible in D). Note that when $C(D)$ is a torsion group the coefficients x_λ satisfy $0\le x_\lambda\le$ order of λ in $C(D)$. Clearly not all vectors v defined by (2) are of irreducible type.

EXAMPLE 1. Suppose we have $\{\mathbf{Z}_{12},S\}$ with $S=\{1,2,3\}$. The irreducible vectors are as follows:

$\langle\langle 12,0,0\rangle\rangle$	$\langle\langle 0,0,4\rangle\rangle$	$\langle\langle 10,1,0\rangle\rangle$	$\langle\langle 0,3,2\rangle\rangle$
$\langle\langle 8,2,0\rangle\rangle$	$\langle\langle 7,1,1\rangle\rangle$	$\langle\langle 6,3,0\rangle\rangle$	$\langle\langle 5,2,1\rangle\rangle$
$\langle\langle 4,4,0\rangle\rangle$	$\langle\langle 3,3,1\rangle\rangle$	$\langle\langle 2,5,0\rangle\rangle$	$\langle\langle 1.4.1\rangle\rangle$
$\langle\langle 0,6,0\rangle\rangle$	$\langle\langle 4,1,2\rangle\rangle$	$\langle\langle 9,0,1\rangle\rangle$	$\langle\langle 2,2,2\rangle\rangle$
$\langle\langle 6,0,2\rangle\rangle$	$\langle\langle 1,1,3\rangle\rangle$	$\langle\langle 3,0,3\rangle\rangle$.	■

In the following discussion, let $\{C(D), S\}$ be a realizable pair. Set

$$\mathfrak{F} = \{v \mid \Psi((\alpha)) = v \text{ for some } \alpha \in D\} \qquad \text{and}$$

$$\mathfrak{I} = \{v \mid \Psi((\gamma)) = v \text{ for some irreducible } \gamma \text{ of } D\}.$$

Define by + and − the operations of regular vector addition and subtraction on the elements of $\mathfrak{F}$. Notice that the set $\mathfrak{F}$ forms a semigroup under +.

Definition. Let n be a positive integer greater than 1. $\mathfrak{I}$ has *n-$(n-1)$ closure* if and only if $z = v_1 + \cdots + v_n - w_1 - \cdots - w_{n-1}$ is in $\mathfrak{I}$ for all $v_1, \ldots, v_n$ and $w_1, \ldots, w_{n-1}$ in $\mathfrak{I}$ for which $z \in \mathfrak{F}$.

The closure properties are in some sense a generalization of the properties V_n and W_n discussed in Section 1. Notice that a Dedekind domain has 2–1 closure if and only if it has property W_2 (and thus V_2). The following examples illustrate more closely the 2–1 closure property.

Example 2. Let $\{\mathbf{Z}_6, S\}$ be a realizable pair with $S = \{2,3\}$. Then a brief check shows $\mathfrak{I} = \{\langle\langle 3,0\rangle\rangle, \langle\langle 0,2\rangle\rangle\}$ which is easily seen to have 2–1 closure. ■

Example 3. Extended calculations on the set $\mathfrak{I}$ listed in Example 1 show that it has 2–1 closure. ■

Example 4. Let D be a Dedekind domain with realizable pair $\{\mathbf{Z}_{10}, S\}$ and $S = \{1,3\}$. Then

$$\mathfrak{I} = \{\langle\langle 10,0\rangle\rangle, \langle\langle 7,1\rangle\rangle, \langle\langle 4,2\rangle\rangle, \langle\langle 1,3\rangle\rangle, \langle\langle 0,10\rangle\rangle\}.$$

$\mathfrak{I}$ is not 2–1 closed since $\langle\langle 10,0\rangle\rangle + \langle\langle 0,10\rangle\rangle - \langle\langle 7.1\rangle\rangle = \langle\langle 3,9\rangle\rangle \notin \mathfrak{I}$. ■

In the following Lemma we list some properties of n-$(n-1)$ closure and find a connection between the closure properties and properties of element factorization in D.

Lemma 2.4. *Let D be a Dedekind domain with realizable pair $\{C(D), S\}$ and set of irreducible vectors $\mathfrak{I}$. Suppose $n \geq 2$ is a positive integer. Then:*

(1) *If $\mathfrak{I}$ has n-$(n-1)$ closure and $\alpha_1, \ldots, \alpha_n$, $\beta_1, \ldots, \beta_{n-1}$ are irreducibles with $\alpha_1 \cdots \alpha_n = \beta_1 \cdots \beta_{n-1}\gamma$ then γ is irreducible.*

(2) *n-$(n-1)$ closure implies k-$(k-1)$ closure for all positive integers k such that $2 \leq k \leq n$.*

(3) *D is HFD if and only if $\mathfrak{I}$ has n-$(n-1)$ closure for all positive integers $n \geq 2$.*

PROOF. For (1), just apply Ψ to (γ). (2) follows directly from the definition and (3) follows easily from (1). ■

Let us consider further how a product of n irreducibles can be factored in a Dedekind domain.

DEFINITION. Let D be an atomic integral domain and n any positive integer. If

$$\mathcal{W}(n) = \{\gamma_1 \cdots \gamma_n \mid \gamma_i \text{ irreducible for } 1 \leq i \leq n \text{ in } D\} \quad \text{and}$$

$$\mathcal{V}(n) = \{m \mid \text{there exist irreducibles } \beta_1, \ldots, \beta_m \text{ such that } \beta_1 \cdots \beta_m \in \mathcal{W}(n)\}$$
$$= \{m \mid \mathcal{W}(n) \cap \mathcal{W}(m) \neq \varnothing\},$$

then define

$$\Phi(n) = |\mathcal{V}(n)| \quad \text{and}$$
$$\Phi(D) = \sup\{\Phi(n) \mid n \text{ a positive integer}\}.$$

In other words, $\Phi(n)$ represents the largest number of factorizations of different lengths that a product of n irreducibles might have in D. At the conclusion of this section we will establish that either $\Phi(D) = 1$ or $\Phi(D) = \infty$. We list some of the basic properties of the Φ-function in the following Lemma.

LEMMA 2.5. *Let D be any atomic integral domain and m and n positive integers with $m \geq n$. Then:*

(1) $\Phi(1) = 1$.
(2) $\Phi(m) \geq \Phi(n)$.
(3) $\Phi(m) = 1$ *implies* $\Phi(n) = 1$.
(4) *D is HFD if and only if* $\Phi(D) = 1$. ■

Clearly in an atomic domain it is possible for $\Phi(n) = \infty$ for some positive integer $n > 1$. To see this, let D be any atomic domain without the ascending chain condition on principal ideals. Let $(\alpha_1) \subset (\alpha_2) \subset \cdots$ be a strictly ascending sequence of principal ideals in D. For all i let δ_i be the nonunit such that $\alpha_i = \delta_i \alpha_{i+1}$. If α_1 can be factored as a product of n irreducibles then the equality $\alpha_1 = \delta_1 \cdot \delta_2 \cdots \delta_k \cdot \alpha_{k+1}$ implies that $\Phi(n)$ is not bounded. Consider now the situation where D is a Dedekind domain with finite class group. Let n be any positive integer. Since there are a finite number of irreducible vectors v in $\mathcal{G}$, there exist positive integers m and r such that m is the maximum number of prime ideals which appear in a factorization of a principal ideal generated by an irreducible of D and r is the minimum number which appears. Thus the longest possible factorization of a product of n irreducibles is less than or equal to nm/r and $\Phi(n)$ is bounded above. If the class group of D is not finite then the last result fails as the next example shows.

EXAMPLE 5. Let $G = \sum_{n=3}^{\infty} \mathbf{Z}_n$ and $S = \{v_m, w_m\}_{m=3}^{\infty}$ where $v_m = (0, \ldots, 1, \ldots)$ and $w_m = (0, \ldots, m-1, \ldots)$ with the nonzero value in each element falling in the $(m-2)$th coordinate. $\{G, S\}$ is a realizable pair by Corollary 1.6 in [13]. Let D be any Dedekind domain associated to $\{G, S\}$. For each $m \geq 3$ let P_m be a prime ideal of class v_m and Q_m a prime ideal of class w_m. Let $P_m Q_m = (\gamma_m)$, $(P_m)^m = (\alpha_m)$ and $(Q_m)^m = (\beta_m)$ for irreducibles γ_m, α_m and β_m of D. Then $(P_m Q_m)^m = (P_m)^m (Q_m)^m$ and $\alpha_m \beta_m = \gamma_m^m$. Thus each $\alpha_m \beta_m \in \mathcal{W}(2)$ and each $\gamma_m^m \in \mathcal{W}(2)$. Hence $m \in \mathcal{V}(2)$ and $\mathcal{V}(2) = \{2, 3, 4, 5, \ldots\}$ implies that $\Phi(2) = \infty$. ∎

EXAMPLE 6. Let D be the ring of integers in a finite extension of the rationals with class number 3. We compute $\Phi(n)$ for any positive integer $n \geq 2$. We first find an upper bound for the lengths of possible factorizations of the product of n irreducibles. Let n be an odd positive integer greater than one. The principal ideal generated by the product of n nonprime irreducibles contains at most $3n$ prime ideals and the principal ideal generated by the product of $(3n+1)/2$ nonprime irreducibles contains at least $3n+1$ prime ideals. Thus no product of n nonprime irreducibles factors as a product of $(3n+1)/2$ irreducibles. Now, let P be a prime ideal of class 1 and Q a prime ideal of class 2. Let α, β and ζ be irreducibles such that $(\alpha) = P^3$, $(\beta) = Q^3$ and $(\zeta) = PQ$. If γ is any irreducible of D then $\alpha^{(n-1)/2} \beta^{(n-1)/2} \gamma = u \zeta^{3(n-1)/2} \gamma$ with u a unit of D and n irreducibles of D can be factored as $(3n-1)/2$ irreducibles. Thus $(3n-1)/2$ is the least upper bound of the length of a factorization when n is odd. Using a similar argument we get a least upper bound of $3n/2$ when n is even.

For the greatest lower bound on the length of factorizations of a product of n irreducibles we consider three cases, $n \equiv 0$, 1 and 2 (mod 3). If $n \equiv 0$ (mod 3) then $\zeta^n = u \alpha^k \beta^k$ with $n = 3k$ and u a unit of D. Thus n irreducibles can be factored as $2n/3$. Since the product of n nonprime irreducibles contains at least $2n$ prime ideals and the product of $(2n-3)/3$ nonprime irreducibles contains at most $2n-3$ prime ideals, n irreducibles cannot be factored as $(2n-3)/3$ irreducibles. Thus $2n/3$ is the greatest lower bound for the number of factors possible for a product of n irreducibles. Using similar reasoning in the other two cases yields $(2n+1)/3$ for $n \equiv 1$ (mod 3) and $(2n+2)/3$ for $n \equiv 2$ (mod 3).

For a given value of n there exists a factorization of n irreducibles into a product of k irreducibles for any k between the least upper and greatest lower bounds. To see this we use the product $u\alpha\beta = \zeta^3$ constructed above. Assume n is an odd positive integer. We show there exists a product of n irreducibles which has a factorization into a product of k irreducibles for each k such that $n \leq k \leq (3n-1)/2$. By a previous argument $\alpha^{(n-1)/2}\beta^{(n-1)/2}\gamma = u\zeta^{3(n-1)/2}\gamma$ with γ any irreducible of D and u a unit. Notice that by repeatedly exchanging the product $\alpha\beta$ for $u\zeta^3$ in $\alpha^{(n-1)/2}\beta^{(n-1)/2}\gamma$, the length of the factorization can be increased by 1 until the product $u\zeta^{3(n-1)/2}\gamma$ is reached. A similar argument works for factorizations of length k when n is even and $n \leq k \leq 3n/2$. For values of k less than n the argument breaks into 3 parts. If $n \equiv 0 \pmod 3$ then by repeatedly exchanging $u\zeta^3$ for $\alpha\beta$ in $u\alpha^{n/3}\beta^{n/3}$, the length of the factorization can again be increased by 1 until $u\zeta^n$ is reached. Thus there exists a factorization of a product of n irreducibles into a product of k irreducibles for each k such that $2n/3 \leq k \leq n$. Similar arguments work when $n \equiv 1$ or $2 \pmod 3$. By subtracting the upper and lower bounds and adding 1 we obtain the following answers mod 6 for $\Phi(n) =$

(1) $(5n+6)/6$ if $n \equiv 0 \pmod 6$,
(2) $(5n+1)/6$ if $n \equiv 1 \pmod 6$,
(3) $(5n+2)/6$ if $n \equiv 2 \pmod 6$,
(4) $(5n+3)/6$ if $n \equiv 3 \pmod 6$,
(5) $(5n+4)/6$ if $n \equiv 4 \pmod 6$,
(6) $(5n-1)/6$ if $n \equiv 5 \pmod 6$. ■

Using Lemma 2.4 we can now deduce the following relationship between the closure properties and the Φ-function.

Corollary 2.6. *Let D be a Dedekind domain with realizable pair $\{C(D), S\}$ and set of irreducible vectors $\mathcal{I}$. Suppose $n \geq 2$ is some positive integer. Then $\mathcal{I}$ has n-$(n-1)$ closure if and only if $\Phi(n) = 1$.*

Proof. Suppose $\alpha_1 \cdots \alpha_n$ is a product of n irreducibles and $\beta_1 \cdots \beta_m$ is a product of m irreducibles for $m \geq n$. If $\alpha_1 \cdots \alpha_n = \beta_1 \cdots \beta_m$ then $\alpha_1 \cdots \alpha_n = \beta_1 \cdots \beta_{n-1}(\beta_n \cdots \beta_m)$ and Lemma 2.4 yields $\beta_n \cdots \beta_m$ is irreducible; hence $n = m$. There is no loss in assuming $m \geq n$ for if $m < n$ the factorization $\alpha_1 \cdots \alpha_n = \beta_1 \cdots \beta_m$ yields

$$\alpha_1 \cdots \alpha_n \alpha_1 \cdots \alpha_{n-m} = \beta_1 \cdots \beta_m \alpha_1 \cdots \alpha_{n-m}$$

and the term on the right has n factors. Reasoning as before $n + n - m = n$ so $n = m$. Similar reasoning supplies the converse. ■

Let D be a Dedekind domain with finite class group in which each ideal class contains a prime ideal. Using the result of Czogala [7] concerning the property V_2, mentioned in Section I, and Corollary 2.6, we see that 2–1 closure in D implies n–$(n-1)$ closure for all positive integers $n \geq 2$. This leads to the following characterization of such domains, the proof of which is an easy application of Theorem 7 in [2], Lemma 2.5 and Corollary 2.6.

THEOREM 2.7. *Let D be a Dedekind domain with finite class group in which each ideal class contains a prime ideal. The following are equivalent:*

(1) *D has class number less than or equal to two.*

(2) *D is HFD.*

(3) *D is CHFD for some $r > 1$.*

(4) $\Phi(2) = 1$.

(5) *$\Phi(n) = 1$ for all $n > 1$.*

(6) *D has 2–1 closure.*

(7) *D has n–$(n-1)$ for all $n \geq 2$.* ∎

Notice that the Theorem above asserts that $\Phi(2) = 1$ implies that D is HFD. After introducing some terminology we will demonstrate that this result does not extend to arbitrary Dedekind domains.

It will often be convenient to refer to a product of a fixed number of prime ideals from a given ideal class. If it is understood that a prime ideal P lies in a given ideal class, say s, of $C(D)$, let $P^{(x)}$ represent the product of x not necessarily distinct prime ideals from the class s. Thus, for the ideal J of equation (1) we can create a representation of the form

$$P_{\alpha_1}^{(x_1)} \cdots P_{\alpha_{kJ}}^{(x_{kJ})}. \tag{3}$$

While for an element α of D the coefficients x_i above are merely the nonzero entries of the vector v associated to (α) we will frequently find such products clearer and more useful than the straight vector notation. For a given ideal I of D we will refer to its representation (3) as the *class type* of I. We will use the symbol $[I]$ to refer to the class type of the ideal I. If γ is irreducible we shall refer to $[(\gamma)]$ as an *irreducible class type.* We will illustrate this terminology in the next example.

EXAMPLE 7. Let n be a positive integer with $n \geq 2$. Let G be the direct sum of n copies of $\mathbf{Z}_{n+2}$ and let D_n be a Dedekind domain with realizable pair $\{G, S\}$ with

$$S = \{(1,0,\ldots,0),\ldots,(0,\ldots,0,1),(n+1,\ n+1,\ldots,n+1)\}.$$

In this case we show $\Phi(n) = 1$ (hence $\Phi(k) = 1$ for $1 \leq k \leq n$) but $\Phi(n+1) \neq 1$.

To begin, notice that the irreducible class types of D_n are

(1) $P_1^{(n+2)}, P_2^{(n+2)}, \ldots, P_n^{(n+2)}$,

(2) $Q^{(n+2)}$,

(3) $P_1^{(1)} P_2^{(1)} \cdots P_n^{(1)} Q^{(1)}$,

where P_i represents a prime ideal taken from the class represented by the ith canonical basis element of G and Q from the remaining class $(n+1, n+1, \ldots, n+1)$ of S. Suppose $\gamma_1 \cdots \gamma_n$ is a product of n nonprime irreducibles of D_n. We consider two cases.

Case 1: Suppose all of the irreducibles in $\gamma_1 \cdots \gamma_n$ are of types (1) or (2). By the pigeon hole principle at least one of the class types $P_1^{(n+2)}, \ldots, P_n^{(n+2)}$, $Q^{(n+2)}$ does not appear in $[(\gamma_1 \cdots \gamma_n)]$. Thus no other factorization of the element $\gamma_1 \cdots \gamma_n$ contains an irreducible of type (3). By a counting argument on the number of prime ideals which appear in $(\gamma_1 \cdots \gamma_n)$, any irreducible factorization of $\gamma_1 \cdots \gamma_n$ will have length n.

Case 2: Suppose exactly k of $\gamma_1 \cdots \gamma_n$ are of type (3). Again by a counting argument, one of the prime ideals $P_1, \ldots, P_n$, Q appears in $[(\gamma_1 \cdots \gamma_n)]$ to a power less than $n + 2$. Thus any irreducible factorization of $\gamma_1 \cdots \gamma_n$ must contain exactly k irreducible factors of type (3). Applying Case 1 to the prime ideals left after k primes of each class are removed from the prime factorization of $(\gamma_1 \cdots \gamma_n)$ yields the desired result. This establishes $\Phi(n) = 1$.

D_n does not have $\Phi(n+1) = 1$ since if $P_1^{(n+2)} = (\gamma_1), \ldots, P_n^{(n+2)} = (\gamma_n)$, $Q^{(n+2)} = (\gamma_{n+1})$ and $P_1^{(1)} \cdots P_n^{(1)} Q^{(1)} = (\beta)$ then $\gamma_1 \cdots \gamma_n \gamma_{n+1} = u\beta^{n+2}$ for u a unit of D and thus $n + 1$ irreducibles can be factored as a product of $n + 2$ irreducibles giving $\Phi(n+2) \geq 2$. ■

Note that the domain D_n is also an example of a Dedekind domain with n-$(n-1)$ closure but not $(n+1)$-n closure. While 2–1 closure does not imply n-$(n-1)$ closure for all $n \geq 2$, we do have the following general criteria for Dedekind domains.

THEOREM 2.8. *Let D be a Dedekind domain with realizable pair $\{C(D), S\}$. The following are equivalent:*

(1) *D is HFD.*

(2) $\Phi(n) = 1$ *for all positive integers n.*

(3) $\mathcal{I}$ *has n-$(n-1)$ closure for all $n \geq 2$.*

(4) *There exists a fixed positive integer m such that $\mathcal{I}$ has n-$(n-1)$ closure for all $n \geq m$.*

(5) $\mathcal{I}$ *has n-$(n-1)$ closure for infinitely many positive integers n.*

(6) $\Phi(n) = 1$ *for infinitely many positive integers n.*

PROOF. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) by Lemma 2.5 part (4) and Corollary 2.6. (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (6) are trivial. (6) $\Rightarrow$ (2) by Lemma 2.5 part (3). ∎

We now demonstrate a property of the Φ-function as applied to atomic domains.

THEOREM 2.9. *Let D be an atomic domain. If $\Phi(n) > 1$ then $\Phi(kn) \geq k + 1$ for all positive integers k. It follows that either $\Phi(D) = 1$ or $\Phi(D) = \infty$, hence D is HFD if and only if $\Phi(D)$ is finite.*

PROOF. Suppose $\Phi(D) \neq 1$. Then there exists a positive integer n such that $\Phi(n) > 1$. Thus there exist irreducibles $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_m$ such that $\alpha_1 \cdots \alpha_n = \beta_1 \cdots \beta_m$ with $n \neq m$. Let k and t be integers such that $0 \leq t \leq k$ and consider the equality

$$(\alpha_1 \cdots \alpha_n)^k = (\beta_1 \cdots \beta_m)^t (\alpha_1 \cdots \alpha_n)^{k-t}. \tag{4}$$

One side of (4) has nk irreducibles and the other $n(k - t) + tm$. We claim that the numbers $\{n(k - t) + tm\}_{t=0}^{k}$ are distinct. To see this suppose that for x and y

$$n(k - x) + xm = n(k - y) + ym.$$

Then clearly $(m - n)x = (m - n)y$ and since $m \neq n$, $x = y$. Thus $\Phi(kn) \geq k + 1$ and $\Phi(D)$ is not bounded. ∎

III. Some results on Dedekind half factorial domains

We use the results of the second section to show that certain arithmetic conditions on the set $\mathcal{I}$ force D to be HFD when $C(D)$ is cyclic. Our work here expands results of Zaks in [41] and [42] and Skula in [36]. We begin by considering a condition which is applicable to any realizable pair $\{C(D), S\}$ but later restrict our attention to pairs where $C(D)$ is cyclic and the elements of S have a certain divisibility property.

DEFINITION. Let D be any Dedekind domain with realizable pair $\{C(D), S\}$ and set of irreducible vectors $\mathcal{I}$. A vector $v = \langle\langle x_s \rangle\rangle_{s \in S}$ of $\mathcal{I}$ is *pure irreducible* if only one of the x_s's is nonzero. If $v \in \mathcal{I}$ is not pure irreducible, then we shall refer to it as *mixed irreducible.*

Notice that if v is a pure irreducible vector of $\mathcal{I}$ then the nonzero x_s must have value $|s|$ in $C(D)$. Also, if $C(D)$ is a torsion group, it is clear that $\mathcal{I}$ will contain $|S|$ different pure irreducible vectors associated to nonprime irreducibles. If these irreducible vectors are the only vectors in $\mathcal{I}$ we have the following.

THEOREM 3.1. *Let $\{C(D), S\}$ be a realizable pair. If the set of irreducible vectors contains only pure irreducible vectors, then D is HFD.*

PROOF. Suppose $\mathcal{I}$ satisfies the hypothesis and let n be any positive integer. Let $v_1, \ldots, v_n$, $w_1, \ldots, w_{n-1}$ be vectors in $\mathcal{I}$. Thus each vector is of the form $\langle\langle 0, \ldots, 0, r_i, 0, \ldots, 0 \rangle\rangle$ for some $1 \leq i \leq k$. If $v_1 + \cdots + v_n - w_1 - \cdots - w_{n-1}$ has only nonnegative entries then we can assume with the appropriate indexing that $w_1 = v_1, \ldots, w_{n-1} = v_{n-1}$ and

$$v_1 + \cdots + v_n - w_1 - \cdots - w_{n-1} = v_n.$$

Thus $\mathcal{I}$ has n-$(n-1)$ closure and D is HFD by Lemma 2.4. This result also follows from fairly straightforward reasoning on the prime ideal factorization of elements. ∎

Note that the above proof does not rely on any properties of $C(D)$ or that S is finite.

EXAMPLE 8. Let n be a positive integer such that $n = pq$ where p and q are distinct primes. Let $\{\mathbf{Z}_n, S\}$ be a realizable pair with $S = \{p, q\}$. Then $\mathcal{I}$ contains no mixed irreducible vectors. To see this, suppose $x_1 p + x_2 q = n$ for $x_1 < q$ and $x_2 < p$. Then p divides x_2 and it follows that $x_2 = 0$ and $x_1 = q$, a contradiction. Thus D is HFD. ∎

EXAMPLE 9. Let $\{\mathbf{Z}_{30}, S\}$ be a realizable pair with $S = \{6, 10, 15\}$. By considering all multiples $n_1 \cdot 6 + n_2 \cdot 10 + n_3 \cdot 15$ with $0 \leq n_1 < 5$, $0 \leq n_2 < 3$ and $0 \leq n_3 < 2$, it is seen that the set $\mathcal{I}$ has no mixed irreducible vectors. Thus by Theorem 3.1 D is HFD. ∎

Example 9 provides a counterexample to a statement made by Steffan in [39]. In this paper Steffan shows that for any Dedekind domain D there exists a constant q such that if $\alpha_1 \cdots \alpha_r = \beta_1 \cdots \beta_s$ are two irreducible factorizations of an element of D then $1/q \leq r/s < q$. In Proposition 3 Steffan claims that when $C(D)$ is the direct sum of cyclic groups generated by the elements of S the smallest possible value of q which satisfies the above inequality is $m/2$, where m is the greatest order in $C(D)$ of an element of S. His example directly following Proposition 3 is the realizable pair presented in Example 9. By his result 5/2 is the minimal value of q for this domain. Although 5/2 is an acceptable value of q, Theorem 3.1 indicates that $q = 1$ also works.

Using the same technique as illustrated above we can generalize the last two examples. Let $\langle s_1, \ldots, s_k \rangle$ represent the subgroup of $\mathbf{Z}_n$ generated by $s_1, \ldots, s_k$.

THEOREM 3.2. *Let $\{\mathbf{Z}_n, S\}$ be a realizable pair with $S = \{s_1, \ldots, s_k\}$ such that*

$$\langle s_i \rangle \cap \langle s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_k \rangle = 0 \qquad \text{for } 1 \le i \le k$$

(i.e. $-\mathbf{Z}_n \cong \mathbf{Z}_{n_1} \oplus \cdots \oplus \mathbf{Z}_{n_k}$). Then D is HFD.

PROOF. Let v be an irreducible vector in $\mathfrak{I}$. If v is not pure then one of the intersections listed above is not empty. Thus all irreducible vectors are pure. ■

We now consider two properties on realizable pairs $\{C(D), S\}$.

DEFINITION. Let $\{\mathbf{Z}_n, S\}$ be a realizable pair.

(1) If all the elements of S divide n, then S has the *all divisor property* $(\mathfrak{AD})$.

(2) If the set S contains 1 then S is said to be *unitary*.

The results of Zaks [41][42] and Skula [36] indicate that characterizing unitary sets S such that $\{\mathbf{Z}_n, S\}$ is HFD depends on considering unitary sets with property $(\mathfrak{AD})$. We now focus on extending these results. To begin, the following lemma is apparent for all realizable pairs $\{\mathbf{Z}_n, S\}$.

LEMMA 3.3. *Suppose $\{\mathbf{Z}_n, S\}$ is a realizable pair with $S = \{s_1, \ldots, s_k\}$ and $n \ge 2$. If $v = \langle\langle x_1, \ldots, x_k \rangle\rangle \in \mathfrak{F}$ then*

$$\sum_{i=1}^{k} x_i s_i = dn \tag{5}$$

for some $d \ge 1$ where each sum and product in (5) is taken over $\mathbf{Z}$. ■

We now give a complete description of $\mathfrak{I}$ as defined in the previous section when S has property $(\mathfrak{AD})$. Given the same setting as Lemma 3.3, let

$$\mathfrak{J} = \left\{ v \,|\, v \in \mathfrak{F} \text{ and } \sum_{i=1}^{k} x_i s_i = n \right\}.$$

It is easily observed that $\mathfrak{F} \supseteq \mathfrak{I} \supseteq \mathfrak{J}$. When the latter two are equal we show that D is HFD.

THEOREM 3.4. *If $\mathfrak{I} = \mathfrak{J}$ then D is HFD.*

PROOF. Suppose $m \in \mathbf{Z}^+$ with $m \ge 2$ and let $v_i = \langle\langle x_{1,i}, \ldots, x_{k,i} \rangle\rangle$ and $w_j = \langle\langle z_{1,j}, \ldots, z_{k,j} \rangle\rangle$ be elements of $\mathfrak{I}$ for $1 \le i \le m$ and $1 \le j \le m-1$. Thus $\sum_{q=1}^{k} x_{q,i} r_q = \sum_{q=1}^{k} z_{q,j} r_q = n$ for all i and j. Suppose $\sum_{t=1}^{m} v_t - \sum_{u=1}^{m-1} w_u \in \mathfrak{F}$. Then

$$\sum_{t=1}^{m} v_t - \sum_{u=1}^{m-1} w_u = \left(\sum_{t=1}^{m} x_{d,t} - \sum_{u=1}^{m-1} z_{d,u}\right)_{d=1}^{k}$$

and

$$\begin{aligned}\sum_{d=1}^{k}\left(\sum_{t=1}^{m} x_{d,t} - \sum_{u=1}^{m-1} z_{d,u}\right) &= \sum_{d=1}^{k}\left(\sum_{t=1}^{m} x_{d,t}\right) - \sum_{d=1}^{k}\left(\sum_{u=1}^{m-1} z_{d,u}\right)\\ &= \sum_{t=1}^{m}\left(\sum_{d=1}^{k} x_{d,t}\right) - \sum_{u=1}^{m-1}\left(\sum_{d=1}^{k} z_{d,u}\right)\\ &= \sum_{t=1}^{m} n - \sum_{u=1}^{m-1} n = mn - (m-1)n = n.\end{aligned}$$

Thus $\sum_{t=1}^{m} v_t - \sum_{u=1}^{m-1} w_u \in \mathcal{S}$ and since $\mathcal{S}$ has m-$(m-1)$ closure for all m, D is HFD by Lemma 2.4. ■

We show by example that the converse of Theorem 3.4 does not hold in general.

Example 10. Let $\{\mathbf{Z}_{24}, S\}$ be a realizable pair with $S = \{4,9\}$. Then $\mathcal{S} = \{\langle\langle 6,0\rangle\rangle, \langle\langle 0,8\rangle\rangle, \langle\langle 3,4\rangle\rangle\}$ and clearly $\mathcal{S} \neq \mathscr{S}$. We claim that D is HFD. Suppose $\gamma_1 \cdots \gamma_s = \beta_1 \cdots \beta_t$ are two irreducible factorizations of an element in D. Considering the irreducibles above as vectors in $\mathcal{S}$, suppose $\gamma_1 \cdots \gamma_s$ contains s_1 vectors of type $\langle\langle 6,0\rangle\rangle$, s_2 vectors of type $\langle\langle 0,8\rangle\rangle$, and s_3 vectors of type $\langle\langle 3,4\rangle\rangle$. Let t_1, t_2 and t_3 be the corresponding numbers of vectors for $\beta_1 \cdots \beta_t$. By counting prime ideals we have $2t_1 + t_3 = 2s_1 + s_3$ and $2t_2 + t_3 = 2s_2 + s_3$. Adding these equalities yields $t = t_1 + t_2 + t_3 = s_1 + s_2 + s_3 = s$ giving D is HFD. ■

However, under property $(\mathfrak{AD})$ the converse of Theorem 3.4 does hold.

Theorem 3.5. *Let $\{\mathbf{Z}, S\}$ be a realizable pair with $S = \{s_1, \ldots, s_k\}$. If S has property $(\mathfrak{AD})$ and D is HFD then $\mathcal{S} = \mathscr{S}$.*

Proof. Suppose $v = \langle\langle x_1, \ldots, x_k\rangle\rangle \in \mathcal{S}$ with $\sum_{i=1}^{k} x_i s_i = tn$ for $t \geq 1$. Let $v' = \langle\langle m_i - x_i\rangle\rangle_{i=1}^{k}$ where m_i is the order of s_i in $\mathbf{Z}_n$. Suppose that γ is an irreducible associated to v and γ' an element associated to v'. Notice that $\overline{\gamma\gamma'} = \prod_{i=1}^{k} \bar{P}_i^{m_i}$ with P_i a prime ideal of class s_i and thus $\gamma\gamma' = \delta_1 \cdots \delta_k$ with each δ_i irreducible. Since D is HFD, this implies γ' is the product of $k-1$ irreducibles, say $\eta_1 \cdots \eta_{k-1}$. Notice that if $v_i = \langle\langle y_{i,1}, y_{i,2}, \ldots, y_{i,k}\rangle\rangle$ is the vector associated with η_i then $\sum_{j=1}^{k} y_{i,j}\cdot s_j \geq n$ for each $1 \leq i \leq k-1$ and thus $\sum_{i=1}^{k}(m_i - x_i)s_i \geq (k-1)n$. Now

$$\sum_{i=1}^{k}(m_i - x_i)s_i = kn - tn = (k-t)n$$

implies that $(k-t)n \geq (k-1)n$ and so $t \leq 1$. Clearly $t > 0$ so $t = 1$. ■

Thus, under property ($\mathfrak{AD}$), Lemma 3.3 can be immediately improved to

Corollary 3.6. *Suppose $\{\mathbf{Z}_n, S\}$ is a realizable pair for $n \geq 2$ and $S = \{s_1, \ldots, s_k\}$ has property ($\mathfrak{AD}$). Then D is HFD if and only if for all irreducible vectors $v = \langle\langle x_1, \ldots, x_k\rangle\rangle$, $\sum_{i=1}^{k} x_i s_i = n$.*

Notice that for cyclic class groups Corollary 3.6 offers a slightly different characterization of splittability. In fact, since Examples 1 and 2 satisfy the Corollary, Zaks' results indicate that the sets $\{12,6,4\}$ and $\{3,2\}$ are splittable. In the following example, we will use the Corollary and show that a given set of positive integers is not splittable as well as produce a unitary set with property ($\mathfrak{AD}$) such that $\mathcal{I} \neq \mathfrak{J}$.

Example 11. Let $\{\mathbf{Z}_{30}, S\}$ be a realizable pair with $S = \{1,6,10,15\}$. S has property ($\mathfrak{AD}$) and is unitary. Since the equation $x_1'\cdot 1 + x_2'\cdot 6 + x_3'\cdot 10 + x_4'\cdot 15 = 30$ has no integer solutions for $0 \leq x_1' \leq 1$, $0 \leq x_2' \leq 4$, $0 \leq x_3' \leq 2$ and $0 \leq x_4' \leq 1$ the vector $v = \langle\langle 1,4,2,1\rangle\rangle$ is in $\mathcal{I}$. Thus, by Theorem 3.5, D is not HFD. Further, by Zaks' splittability criteria the set $\{30,5,3,2\}$ is not splittable. ■

In general, suppose the hypothesis of Corollary 3.6 holds and that $v = \langle\langle x_i\rangle\rangle_{i=1}^{k}$ is an element of $\mathfrak{F}$ with $\sum_{i=1}^{k} x_i s_i = dn$ for $d > 1$. If there exist nonnegative integers x_i' such that $\sum_{j=1}^{k} x_j' s_j = n$ with $x_j' \leq x_j$ then we shall say that the equation $\sum_{i=1}^{k} x_i s_i = dn$ has a *subsum* which sums to n.

The next example gives a nonunitary set with property ($\mathfrak{AD}$) which also has $\mathcal{I} \neq \mathfrak{J}$.

Example 12. Let $\{\mathbf{Z}_{210}, S\}$ be a realizable pair with $S = \{2,7,42,70,105\}$. Note that S has property ($\mathfrak{AD}$). Considering all subsums of $0\cdot 2 + 1\cdot 7 + 3\cdot 42 +$

$2 \cdot 70 + 1 \cdot 105 = 420$ shows that $v = \langle\langle 0,1,3,2,1 \rangle\rangle$ is irreducible in $\mathfrak{I}$. Thus, as above, D is not HFD and $\{110,30,5,3,2\}$ is not splittable. ■

As we further consider conditions on the set S which force $\{\mathbf{Z}_n, S\}$ to be HFD, the following lemma, a form of which appears in [41], will be useful.

LEMMA 3.7. *Suppose $\{\mathbf{Z}_n, S\}$ is a realizable pair and S has property* $(\mathfrak{A}\mathfrak{D})$. *If $\{\mathbf{Z}_n, S\}$ is HFD and T is a subset of S such that T generates $\mathbf{Z}_n$ then $\{\mathbf{Z}_n, T\}$ is also HFD.*

PROOF. In the presence of the hypothesis, Corollary 3.6 gives a condition for irreducible which is inherited by subsets. ■

In [41] Zaks show that if $\{\mathbf{Z}_n, S\}$ is a realizable pair with associated Dedekind domain D and S is unitary then D HFD implies that S has property $(\mathfrak{A}\mathfrak{D})$. We now produce a partial converse to this result.

THEOREM 3.8. *Let $\{\mathbf{Z}_n, S\}$ be a realizable pair with $S = \{1,r,s\}$. D is HFD if and only if S has property* $(\mathfrak{A}\mathfrak{D})$.

PROOF. Because of Zaks' result we are only concerned about one implication. Suppose $v = \langle\langle x_1, x_2, x_3 \rangle\rangle$ is an irreducible vector of $\mathfrak{I}$. Suppose $x_1 + x_2 r + x_3 s = dn$ for $d > 1$. We consider two cases: (1) Suppose $x_1 + x_2 r < n$. Then $x_3 s > n$ and since s divides n, setting $x_1' = 0$, $x_2' = 0$ and $x_3' = n/s$, we have $x_1' + x_2' r + x_3' s = n$. Hence v is not irreducible. (2) Suppose $x_1 + x_2 r > n$. If $x_2 r > n$ then the result follows in a manner similar to (1). If $x_2 r < n$ then $x_1 > n - x_2 r$. Set $x_1' = n - x_2 r$ and $x_2' = x_2$. Then $x_1' + x_2' r = n$ and again v is not irreducible. Thus $\mathfrak{I} = \mathfrak{J}$ and by Theorem 3.4 D is HFD. ■

Example 11 shows that the last theorem is the best result we can hope for in terms of the number of elements in S. The next corollary follows from Lemma 3.7 and Theorem 3.8.

COROLLARY 3.9. *Let $\{\mathbf{Z}_n, S\}$ be a realizable pair. Then:*
(1) *If $S = \{1,s\}$ then D is HFD if and only if S has property* $(\mathfrak{A}\mathfrak{D})$.
(2) *If $S = \{r,s\}$ and has property* $(\mathfrak{A}\mathfrak{D})$ *then D is HFD.* ■

Example 10 indicates that the converse of (2) does not hold in general.

EXAMPLE 13. Let n_1 and n_2 be any two consecutive positive integers and let n be any positive integer divisible by both n_1 and n_2. If D is a Dedekind domain associated with the realizable pair $\{\mathbf{Z}_n, \{n_1, n_2\}\}$ then D is HFD. ■

Theorem 3.8 implies the following result with respect to splittability.

COROLLARY 3.10. *Let r and s be positive integers not equal to 1. Let m be any common multiple of r and s. Then the set $\{m, m/r, m/s\}$ is splittable.* ■

Since by Lemma 3.7 any subset of a splittable set is splittable, the last corollary implies that the sets $\{m, m/r\}$, $\{m, m/s\}$ and $\{m/r, m/s\}$ are splittable.

The results of Theorem 3.8 raise the following question. If $\{\mathbf{Z}_n, S\}$ is a realizable pair and $S = \{1, r, s\}$ a set with property $(\mathfrak{AD})$, what additional elements divisors of n can be added to S so that the HFD conclusion will remain valid? The following theorem provides a partial answer.

THEOREM 3.11. *Let $\{\mathbf{Z}_n, S\}$ be a realizable pair with $S = \{1, r, s\}$ a set with property $(\mathfrak{AD})$. Suppose t_1 is a positive integer not in S such that $t_1 | r$ or $t_1 | s$ and that $t_1, t_2, \ldots, t_j$ is a sequence of positive integers such that $t_i | t_{i-1}$ for all $2 \leq i \leq j$. If $S' = \{1, r, s, t_1, \ldots, t_j\}$ then any Dedekind domain D associated with the realizable pair $\{\mathbf{Z}_n, S'\}$ is HFD.*

PROOF. Notice that S' has property $(\mathfrak{AD})$. Let D be Dedekind with realizable pair $\{\mathbf{Z}_n, S'\}$ and set of irreducible vectors $\mathfrak{I}$. Let x_1, x_2, x_3 and $y_1, \ldots, y_j$ be nonnegative integers such that $v = \langle\langle x_1, x_2, x_3, y_1, \ldots, y_j \rangle\rangle$ is in $\mathfrak{I}$. Suppose

$$x_1 + x_2 r + x_3 s + y_1 t_1 + \cdots + y_j t_j = dn \qquad \text{for some } d > 1. \tag{6}$$

Since r and s divide n, $x_2 r + x_3 s < 2n$. If $x_2 r + x_3 s < n$ then, since $x_1 + y_1 t_1 + \cdots + y_j t_j > n$ and $t_1 | n$, clearly a subsum of $x_1 + y_1 t_1 + \ldots + y_j t_j$ sums to n. So assume $n < x_2 r + x_3 s < 2n$. Note that

$$2n - (x_2 r - x_3 s) > n - x_2 r \quad \text{and} \quad 2n - (x_2 r + x_3 s) > n - x_3 s.$$

Now, $x_1 + y_1 t_1 + \cdots + y_j t_j \geq 2n - (x_2 r + x_3 s)$. Since $t_1 | r$ or $t_1 | s$, t_1 divides either $n - x_2 r$ or $n - x_3 s$. Thus some subsum of $x_1 + y_1 t_1 + \cdots + y_j t_j$ sums to either $n - x_2 r$ or $n - x_3 s$. Without loss of generality assume $x_1' + y_1' t_1 + \cdots + y_j' t_j = n - x_2 r$ for $x_1' \leq x_1$, $y_1' \leq y_1, \ldots, y_j' \leq y_j$. Thus $(x_1' + y_1 y_1' t_1 + \cdots + y_j' t_j) + x_2 r = n$ and some subsum of (6) sums to n. ■

We close this section with a Corollary which follows from the last Theorem and Lemma 3.7.

COROLLARY 3.12. *Let $\{\mathbf{Z}_n, S\}$ be a realizable pair with $S = \{r, s, t_1, \ldots, t_j\}$ such that $t_1 | r$ or $t_1 | s$ and $t_i | t_{i-1}$ for all $2 \leq i \leq j$. Then*

(1) *If S has property $(\mathfrak{AD})$ then D is HFD.*

(2) *If m is any common multiple of r and s then the set $\{m, m/r, m/s, m/t_1, \ldots, m/t_j\}$ is splittable.* ■

IV. Congruence half factorial domains

We now consider Dedekind domains which are CHFD for some $r > 1$ but not HFD. In [2] we established the following.

THEOREM 4.1. *Let $\{\mathbf{Z}_n, S\}$ be a realizable pair with $S = \{1, n-1\}$ and $n > 3$. Then any Dedekind domain associated with $\{\mathbf{Z}_n, S\}$ is not HFD but is CHFD of order $r = n - 2$.* ■

We build on the above result exploring factorization in Dedekind domains with finite class groups. We begin with the cyclic case.

THEOREM 4.2. *Assume D is a Dedekind domain with realizable pair $\{\mathbf{Z}_n, S\}$ such that 1 and m are in S (there may be prime ideals in other classes). Set* $\mathrm{GCD}(m,n) = k$, *$ks = m$ and $kt = n$. If D is CHFD of order $r > 1$ then r divides $s - 1$.*

PROOF. Let Q be a prime ideal of the class 1 and R a prime ideal of the class m. Then t is the order of m in $\mathbf{Z}_n$. Let α, β, and γ be irreducibles such that

$$(\alpha) = Q^n, \quad (\beta) = R^t, \quad \text{and} \quad (\gamma) = Q^{n-m}R.$$

Thus

$$(\gamma^t) = (Q^{n-m}R)^t = Q^{(t-s)n}R^t = (Q^n)^{t-s}R^t = (\alpha^{t-s}\beta).$$

CHFD of order r then implies $t \equiv t - s + 1 \pmod{r}$. Hence r divides $s = 1$. ■

Notice that if m divides n then $s = 1$ and $s - 1 = 0$ and Theorem 4.2 tells us nothing. The Theorem also provides us an alternate proof Theorem 7 of [2]. Let D be a finite ring of algebraic integers. Then there are prime ideals in 1 and m for all m. Thus CHFD of order r implies r divides $s - 1$ for all possible values of s, which can never occur unless $n \leq 2$.

The next Corollary follows from the above Theorem. We note that (1) is the result established by Zaks [41] mentioned in Section III.

COROLLARY 4.3. *Assume D is a Dedekind domain with realizable pair $\{\mathbf{Z}_n, S\}$ and that 1 and m are in S. Then:*

(1) *If m does not divide n then D is not HFD.*

(2) *If* $\mathrm{GCD}(m,n) = 1$ *then D is not HFD and D CHFD of order r implies that r divides $m - 1$.*

(3) *If $m = n - 1$ then D is not HFD and D CHFD of order r implies that r divides $n - 2$.*

PROOF. For (1), since m does not divide n, $s \neq 1$ so $s - 1 \geq 1$. By Theorem 4.2 there is an upper bound $(s - 1)$ for the values r. Since HFD implies CHFD for all $r > 1$, D is not HFD. The statements (2) and (3) follow from (1) and Theorem 4.2. ■

Let n be an even positive integer. If D is a Dedekind domain with realizable pair $\{\mathbf{Z}_n, S\}$ with $S = \{1,2\}$ or $\{1,n/2\}$, then Corollary 3.9 indicates that D is HFD. We consider similar situations in the next three Theorems when n is odd.

THEOREM 4.4. *Let $n \geq 3$ be an odd positive integer and assume D is a Dedekind domain with realizable pair $\{\mathbf{Z}_n, S\}$ such that there are prime ideals in classes 1 and 2. Then D is not CHFD for any $r > 1$.*

PROOF. If n is odd then $\mathrm{GCD}(2,n) = 1$ and the result holds by Corollary 4.3 part (2). ■

THEOREM 4.5. *Let $n \geq 3$ be an odd positive integer and assume that D is a Dedekind domain with realizable pair $\{\mathbf{Z}_n, S\}$ such that there are prime ideals in the classes determined by 1 and $(n + 1)/2$. Then D is not CHFD for any $r > 1$.*

PROOF. Let R and Q represent prime ideals of the classes $(n + 1)/2$ and 1 respectively. Let Δ_i be an element of D such that $[(\Delta_k)] = R^{(i)}Q^{(x_i)}$ where $0 \leq i \leq n$ and x_i is the smallest positive integer such that

$$\left(\frac{n+1}{2}\right)i + x_i \equiv 0 \pmod{n}.$$

Let v_i be the vector associated with $[(\Delta_i)]$. We wish to identify which vectors $v_1, \ldots, v_n$ are irreducible where $v_i = \langle\langle i, x_i \rangle\rangle$. For $i = 2k$, the least positive residue of $[(n + 1)/2]i$ is k and hence $x_i = n - k$. For odd $i = 2k + 1$, the least positive residue is $[(n + 1)/2] + k$ and hence $x_i = [(n - 1)/2] - k$. Now, notice for $2k$ that $v_{2k} = v_{2k-1} + v_1$ so none of the vectors $v_2, v_4, \ldots, v_{n-1}$ is irreducible. On the other hand, the vectors v_{2k+1} are irreducible since the R "exponents" $1, 3, \ldots, 2k + 1$ are increasing while the Q "exponents" are decreasing. Thus the irreducible vectors are $v_0, v_1, v_3, \ldots, v_{2k+1}, \ldots, v_{n-2}$, and v_n. To show that D is not CHFD of order $r > 1$ choose k_1, k_2, and k_3 so that $y = k_1 + k_2 + k_3 = (n - 3)/2$ (this can be done since the k_i need not be distinct and may be zero). Then

$$(2k_1 + 1) + (2k_2 + 1) + (2k_3 + 1) = 2y + 3 = n.$$

Also

$$[(n-1-2k_1)/2] + [(n-1-2k_2)/2] + [(n-1-2k_3)/2] = n.$$

Thus $v_0 + v_n = v_{2k_1+1} + v_{2k_2+1} + v_{2k_3+1}$ yields 2 irreducibles written as the product of 3 irreducibles. It follows that D is not CHFD for any $r > 1$. ■

THEOREM 4.6. *Let $n \geq 3$ be an odd positive integer and assume D is a Dedekind domain with realizable pair $\{\mathbf{Z}_n, S\}$ with $S = \{1, (n-1)/2\}$. Then D is CHFD of order $(n-3)/2$.*

PROOF. We proceed in a manner similar to Theorem 4.5. Let R be a prime of class $(n-1)/2$ and Q a prime of class 1. Define Δ_i and v_i in an analogous manner to the proof of the last Theorem. For $i = 2k$, the least positive residue of $[(n-1)/2]i$ is $n-k$ and hence $x_i = k$. For odd $i = 2k+1$, the least positive residue is $[(n-1)/2] - k$ and hence $x_i = [(n+1)/2] + k$. Notice that $v_{2k} = \sum_{i=1}^{k} v_2$ and $v_{2k+1} = v_{2k} + v_1$. Thus the only irreducible class types are

$$[\Delta_0] = Q^{(n)}, \quad [\Delta_1] = R^{(1)}Q^{((n+1)/2)}, \quad [\Delta_2] = R^{(2)}Q^{(1)}, \quad [\Delta_n] = R^{(n)}.$$

There are three key identities among the irreducible vectors:

(1) $v_0 + v_n = \left(\sum_{i=1}^{(n-1)/2} v_2\right) + v_1.$

(2) $v_1 + v_n = \sum_{i=1}^{(n+1)/2} v_2.$

(3) $v_0 + v_2 = v_1 + v_1.$

Notice that (1) and (2) show that 2 irreducibles can be factored as the product of $(n+1)/2$ irreducibles. Thus if D has CHFD of order r, then r divides $(n+1)/2 - 2 = (n-3)/2$. Also, any reduction of a product of irreducibles using these identities produces a new product whose number of irreducibles is congruent to the number of irreducibles in the original modulo $(n-3)/2$.

To finish we establish some basic notations. Let elements of the form $\alpha_1, \ldots, \alpha_r$ represent irreducibles of D with vector v_0, $\beta_1, \ldots, \beta_s$ represent irreducibles with vector v_1, $\gamma_1, \ldots, \gamma_t$ represent irreducibles with vector v_2, and $\xi_1, \ldots, \xi_u$ represent irreducibles with vector v_n. We will use (1), (2) and (3) to show that any product of irreducibles can be reduced to a product (modulo $(n-3)/2$)) that involves only two types of irreducible vectors. In fact, an analysis of the reductions show that any factorization can be reduced to one of the following three kinds:

(a) $\beta_1 \cdots \beta_s \gamma_1 \cdots \gamma_t$,
(b) $\alpha_1 \cdots \alpha_r \beta_1 \cdots \beta_s$,
(c) $\gamma_1 \cdots \gamma_t \xi_1 \cdots \xi_u$.

If one has two factorizations of the elements, reduce both to one of the kinds above and then proceed as follows: assume $\beta_1 \cdots \beta_s \gamma_1 \cdots \gamma_t = \beta_1' \cdots \beta_{s'}' \gamma_1' \cdots \gamma_{t'}'$. By counting the number of ideals of class 1 and $(n-1)/2$ on each side of the last factorization we have that $s + 2t = s' + 2t'$ and $[(n+1)/2]s + t = [(n+1)/2]s' + t'$. By the first equation $s - s' = 2(t' - t)$ and by the second $t' - t = [(n+1)/2](s - s')$. Hence $s - s' = [(n+1)/2](s - s')$ so $s = s'$ and it follows that $t = t'$. Therefore $s + t = s' + t'$. The factorizations using $\alpha_1 \cdots \alpha_r \beta_1 \cdots \beta_s$ and $\gamma_1 \cdots \gamma_t \xi_1 \cdots \xi_u$ are argued in a similar fashion. These latter cases are easier, since one of the irreducible class types in each case has prime factors from only one class.

It is left to show that in two factorizations the reductions do not produce different kinds. The arguments are similar to the one above. We include one here. Assume $\beta_1 \cdots \beta_s \gamma_1 \cdots \gamma_t = \alpha_1' \cdots \alpha_{r'}' \beta_1' \cdots \beta_{s'}'$. Again by counting prime ideal of the classes 1 and $(n-1)/2$ we have $s + 2t = s'$ and $[(n+1)/2]s + t = nr' + [(n+1)/2]s'$. Substituting the first equation in the second yields $r' = -t$. Hence $r' = t = 0$ and the factorization reduces to $\beta_1 \cdots \beta_s = \beta_1' \cdots \beta_{s'}'$ where it follows $s = s'$ again by counting prime ideals. The other "mixed" cases have similar reductions to the trivial case. This completes the argument. ∎

Summarizing, if D is Dedekind we now have complete answers for HFD and CHFD when the class group of D is $\mathbf{Z}_n$ for sets S of the form $\{1, m\}$ where m is one of 2, $(n-1)/2$ (n odd), $n/2$ (n even), $(n+1)/2$ (n odd), and $n-1$.

In [2] we show that for Dedekind domains with class number less than or equal to three the conditions CHFD for some $r > 1$ and HFD are identical. Theorem 4.1 shows that for any $n > 3$ a Dedekind domain with class number n can be constructed which is CHFD for some $r > 1$ but not HFD. We explore the relationships between HFD and CHFD for the particular case $n = 4$. The results so far from Theorem 3.1 and Corollary 4.3 imply the following for Dedekind domains with class group $\mathbf{Z}_4$.

COROLLARY 4.7. *Let D be a Dedekind domain with realizable pair* $\{\mathbf{Z}_4, S\}$. *Then D is HFD if and only if* $S = \{1\}$, $S = \{3\}$, $S = \{1,2\}$, *or* $S = \{2,3\}$. ∎

We now consider the case where D has class group K, the Klein 4-group.

THEOREM 4.8. *Let D be a Dedekind domain with class group K the Klein 4-group. The following are equivalent:*

(1) *D is HFD.*

(2) *D is CHFD for all* $r > 1$.

(3) *D is CHFD for some* $r > 1$.

(4) *All of the nonprincipal prime ideals are in at most two classes.*

PROOF. Since (1) and (2) are equivalent and (2) implies (3), we show that (3) implies (4) and that (4) implies (1). By the Grams result, if $\{K, S\}$ is the realizable pair associated with D, then S generates K. Let $K = \{e, a, b, ab\}$. As usual, disregarding the prime ideals which lie in e, we have two cases.

Case 1: $S = \{a, b, ab\}$. The original result of Carlitz [1] shows that in this case D is not HFD. Theorem 7 in [2] shows that D is not CHFD for any $r > 1$. Thus (3) implies (4).

Case 2: $S = \{a, b\}$. It is easy to show that all the irreducibles of D are pure. Thus by Theorem 3.1 D is HFD.

The other possibilities ($S = \{a, ab\}$ and $S = \{b, ab\}$) are equivalent to Case 2 using a notation change. Thus (4) implies (1). ■

We summarize the last two results together with the main result from [2] in the following.

COROLLARY 4.9. *Let D be a Dedekind domain with class number* ≤ 4. *Then D is CHFD for some* $r > 1$ *if and only if D is HFD or D has class group* $\mathbf{Z}_4$ *and* $S = \{1,3\}$ *(in which case D is CHFD of order* 2*).* ■

We continue in a similar manner and will show that a result similar to Theorem 4.8 holds for Dedekind domains with class group $\mathbf{Z}_3 \oplus \mathbf{Z}_3$. In the proof we will require the following Corollary which is a direct consequence of Zaks' splittability criteria.

COROLLARY 4.10. *Let D be a Dedekind domain with class group G, a finite Abelian p-group for some prime integer p. If the principal ideal generated by every nonprime irreducible element of D is a product of p prime ideals then D is HFD.* ■

We now prove the following.

THEOREM 4.11. *Let D be a Dedekind domain with class group* $\mathbf{Z}_3 \oplus \mathbf{Z}_3$. *Then the following are equivalent:*

(1) *D is HFD.*

(2) *D is CHFD for all* $r > 1$.

(3) *D is CHFD for some $r > 1$.*

(4) *S contains no more than 3 elements and either*

(4a) *$S = \{\alpha, \beta\}$ with $\alpha + \beta \neq 0$ or*

(4b) *$S = \{\alpha, \beta, \gamma\}$ with each of $\alpha + \beta$, $\alpha + \gamma$, and $\beta + \gamma$ nonzero and $\alpha + \beta + \gamma = 0$.*

PROOF. As usual (1) implies (2) and (2) implies (3) are evident so we need only establish (3) implies (4) and (4) implies (1).

For (3) implies (4) we note that $\mathbf{Z}_3 \oplus \mathbf{Z}_3$ being noncyclic implies that S must contain at least two elements. We establish first that for distinct elements u, v and w in $\mathbf{Z}_3 \oplus \mathbf{Z}_3$ if $\{u, v\} \subseteq S$ then $u + v \neq 0$ and next that if $\{u, v, w\} \subseteq S$ then $u + v + w = 0$. For the first statement we follow reasoning similar to that in [2] by letting P and Q be any primes in the classes u and v and observing that when $u + v = 0$ we get P^3, Q^3 and PQ all producing irreducibles. Thus $P^3Q^3 = (PQ)^3$ yields a product to two irreducibles written as a product of three irreducibles which contradicts CHFD for some $r > 1$. For the second statement assume $\{u, v, w\} \subseteq S$. By what has been established we have u, v, w, $u + v$, $u + w$ and $v + w$ are all nonzero. We also claim these are distinct. That u is distinct from v, w, $u + v$ and $u + w$ is obvious. We see $u \neq v + w$ for if $u = v + w$ then $2u + v + w = 0$ and setting P, Q and R in the classes v, w and u we have $(PQR^2)^3 = P^3Q^3(R^3)^2$. This yields the product of 3 irreducibles which can be written as a product of 4 irreducibles which again contradicts CHFD for some $r > 1$. Argument of the other possibilities in this same manner yields u, v, w, $u + v$, $u + w$ and $v + w$ are distinct elements of $\mathbf{Z}_3 \oplus \mathbf{Z}_3$. Also none is zero either by assumption or by the first argument given. Since $2u$, $2v$, $2w$ are also nonzero elements and $\mathbf{Z}_3 \oplus \mathbf{Z}_3$ has only eight nonzero elements, it follows that the intersection of $\{2u, 2v, 2w\}$ with $\{u, v, w, u + v, u + w, v + w\}$ is nonempty. Simple calculations yield the only possibilities are $2u = v + w$, $2v = u + w$ or $2w = u + w$. Any of these three produce the desired equation $u + v + w = 0$. To establish (3) implies (4) we observe that if S contains either exactly two elements or exactly three elements, then the above establishes the conditions of (4a) or (4b). We see S cannot contain four elements for $\{\alpha, \beta, \gamma\} \subseteq S$ yields by the reasoning above that $\{\alpha, \beta, \gamma, \alpha + \beta, \alpha + \gamma, \beta + \gamma, 2\alpha + \beta, 2\alpha + \gamma\}$ are all of the nonzero elements. The addition of any of the last five elements to $\{\alpha, \beta, \gamma\}$ contradicts the second property established as is seen, for example, by $\{\alpha, \beta, \gamma, \alpha + \beta\} \subseteq S$ yielding $\{\alpha, \beta, \alpha + \beta\} \subseteq S$, hence $\alpha + \beta + \alpha + \beta = 0$. This gives $\alpha = \beta$. Similar contradictions are obtained replacing $\alpha + \beta$ with any of $\alpha + \gamma$, $\beta + \gamma$, $2\alpha + \beta$ or $2\alpha + \gamma$.

To show that (4) implies (1) we observe in the case (4a) that the only possible irreducible types are P^3, PQ^2, P^2Q, or Q^3 where P and Q belong to the classes α and β (notice that PQ^2 and P^2Q may not produce irreducibles). Thus D is HFD by Corollary 4.10. In the case (4b) the only possible products producing irreducibles (again, not all may give irreducibles) are P^3, Q^3, R^3, PQR, P^2Q, PQ^2, QR^2, Q^2R, PR^2, and P^2R where P, Q and R are from the classes α, β and γ. Again, Corollary 4.10 establishes D is HFD. ∎

It is now easy to show that the cases thus far discussed (that is, when the class group of D is isomorphic to either $\mathbf{Z}_2$, $\mathbf{Z}_3$, $\mathbf{Z}_2 \oplus \mathbf{Z}_2$, or $\mathbf{Z}_3 \oplus \mathbf{Z}_3$) are the only instances where D HFD and D CHFD for some $r > 1$ are equivalent. We begin with a result that follows from the work done in [42] and [2].

THEOREM 4.12. *Let n be a positive integer divisible by either*

(1) *a prime greater than* 3 *or*

(2) 6.

Let G be any finite abelian group of order n. There exists a subset S of $G - \{0\}$ and a positive integer $r > 1$ such that any Dedekind domain associated to the realizable pair $\{G,S\}$ is CHFD of order r but not HFD.

PROOF. If G is cyclic the domain constructed in Theorem 4 of [2] will suffice. Suppose G is not cyclic. Decompose G into a direct sum of cyclic groups, say $G \cong \mathbf{Z}_{n_1} \oplus \mathbf{Z}_{n_2} \oplus \cdots \oplus \mathbf{Z}_{n_k}$. This decomposition can be accomplished so that one of these summands has order greater than 3. Without loss of generality assume n_1 is this summand. Let D be a Dedekind domain with class group G with $S = \{(1,0,\ldots,0),(0,1,\ldots,0),\ldots,(0,0,\ldots,1),(n_1 - 1,0,\ldots,0)\}$. $\{G,S\}$ is a realizable pair by the Grams result. An argument nearly identical to that used in Theorem 4 of [2] shows that D is CHFD of order $n_1 - 2$ but not HFD. ∎

We now consider the cases where $C(D)$ is isomorphic to the direct sum of copies of $\mathbf{Z}_2$ or $\mathbf{Z}_3$.

THEOREM 4.13. *Let c be a positive integer greater than* 2 *and let $\{G,S\}$ be the realizable pair consisting of $S = \{(1,0,\ldots,0),(0,1,\ldots,0),\ldots,(0,0,\ldots,1),$ $(1,1,\ldots,1)\}$ and either*

(1) $G \cong \sum_{i=1}^{c} \mathbf{Z}_2$ *or*

(2) $G \cong \sum_{i=1}^{c} \mathbf{Z}_3$.

Any Dedekind domain D associated with the pair $\{G,S\}$ of either (1) *or* (2) *is not HFD but is CHFD of order $c-1$.*

PROOF. For (1), to see that D is not HFD, let $P_1, P_2, \ldots, Q$ be prime ideals taken from each elment of S in the order in which they appear. Each of the class types $P_1^{(2)}, P_2^{(2)}, \ldots, Q^{(2)}$ and $P_1^{(1)}P_2^{(1)}\cdots Q^{(1)}$ are clearly irreducible. Let $(\gamma_1) = P_1^2$, $(\gamma_2) = P_2^2, \ldots, (\gamma_{c+1}) = Q^2$ and $(\xi) = P_1P_2\cdots Q$. Thus $\gamma_1\gamma_2\cdots\gamma_{c+1} = u\xi^2$ where u is a unit and thus D is not HFD.

We now argue that D is CHFD of order $c-1$. The irreducible types listed above $(P_1^{(2)}, \ldots, Q^{(2)}, P_1^{(1)}P_2^{(1)}\cdots Q^{(1)})$ are the only irreducible types possible since each element of G has order two. We will refer to the irreducibles associated to the class types $P_1^{(2)}, P_2^{(2)}, \ldots, P_c^{(2)}$ as irreducibles of pure basis type, the irreducibles associated to $Q^{(2)}$ as pure of non-basis type, and those associated to $P_1^{(1)}\cdot P_2^{(1)}\cdots Q^{(1)}$ as mixed type. Let γ be any nonzero nonunit of D. If $\gamma = \alpha_1\alpha_2\cdots\alpha_m$ is a factorization of γ into irreducibles of pure basis type, then by an argument similar to that used in the proof of Theorem 3.1 any other factorization of γ into irreducibles will contain m irreducible factors. So suppose γ contains irreducibles of pure non-basis type or of mixed type. Suppose $\gamma = \beta_1\cdots\beta_n = \delta_1\cdots\delta_m$ are two irreducible representations of γ. Now suppose $\gamma = \delta_1\cdots\delta_m$ has

(1) r irreducibles of pure basis type,

(2) t irreducibles of pure non-basis type,

(3) s irreducibles of mixed type.

Let r', t' and s' be the similar parameters for the factorization $\beta_1\cdots\beta_n$. Notice that $t' = t + k$ for some integer k. Using the unique factorization of ideals in D, we must have $s' = s - 2k$ and $r' = r + kc$. Now $m = r + t + s$ and $n = r' + s' + t'$ implies that

$$n - m = t' + s' + r' - (r + t + s) = k - 2k + kc = k(c-1).$$

Thus $n \equiv m \pmod{c-1}$.

For (2), let $P_1, \ldots, P_c$, Q be prime ideals as in the proof of (1). The irreducible class types of D are easily seen to be

(1) $P_i^{(3)}$ for $1 \le i \le c$,

(2) $Q^{(3)}$,

(3) $P_1^{(2)}P_2^{(2)}\cdots P_c^{(2)}Q^{(1)}$,

(4) $P_1^{(1)}P_2^{(1)}\cdots P_c^{(1)}Q^{(2)}$.

We will again refer to irreducibles under (1) as irreducibles of pure basis type, those under (2) as irreducibles of pure non-basis type, and irreducibles under (3) and (4) as irreducibles of mixed type. Continuing in a manner similar to the proof of part (1) of this theorem, let γ be a nonzero nonunit of D and $\gamma = \gamma_1 \cdots \gamma_m$ an irreducible factorization of γ. As before we pass to the case where $\gamma = \gamma_1 \cdots \gamma_m$ is an irreducible factorization which contains irreducibles of pure non-basis type or of mixed type. Suppose $\gamma_1 \cdots \gamma_m$ contains

(1) n_i irreducibles of pure basis type $P_i^{(3)}$,

(2) r irreducibles of pure non-basis type,

(3) m_1 irreducibles of mixed type $P_1^{(2)} P_2^{(2)} \cdots P_c^{(2)} Q^{(1)}$,

(4) m_2 irreducibles of mixed type $P_1^{(1)} P_2^{(1)} \cdots P_c^{(1)} Q^{(2)}$.

If $\gamma = \beta_1 \cdots \beta_n$ is another irreducible factorization of γ let n_i', r', m_1', and m_2' be the corresponding parameters. Set $q_1 = m_1 - m_1'$, $q_2 = m_2 - m_2'$, $k = r - r'$ and $s = n_1 - n_1'$. Simple calculations yield

$$q_2 = \frac{-3k - q_1}{2}, \quad s = \frac{k - q_1}{2} \quad \text{and} \quad n - m = (c - 1)\left(\frac{k - q_1}{2}\right).$$

By counting ideals we can show that if k is even then so too must q_1 and if k is odd then so too must q_1. Thus $k + q_1$ is always even and $c - 1$ divides $n - m$. ∎

We now summarize our results in the following.

COROLLARY 4.14. *Let G be any finite Abelian group. There exists a Dedekind domain with class group G which is not HFD but is CHFD for some $r > 1$ if and only if G is not isomorphic to $\mathbf{Z}_2$, $\mathbf{Z}_3$, $\mathbf{Z}_2 \oplus \mathbf{Z}_2$, or $\mathbf{Z}_3 \oplus \mathbf{Z}_3$.*

PROOF. Due to Theorem 4.12 we need only discuss the case where G has order 2^k or 3^j for k and j positive integers greater than 2. Suppose G is such a group. Again, if G is cyclic we are finished by Theorem 4 in [2]. If G is not cyclic decompose G into a product of cyclic groups. If G contains a summand greater than 3, then the argument offered in Theorem 4.12 again works. If G does not contain a summand greater than 3 then G is isomorphic to one of the groups mentioned in Theorem 4.13. ∎

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